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On a Markov chain roulette-type game

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Abstract
A Markov chain on non-negative integers which arises in a roulette-type game is discussed. The transition probabilities are $p_{01} = \rho$, $p_{Nj} = \delta_{Nj}$, $p_{i,i+W} = q$, $p_{i,i-1} = p = 1 - q$, $1 \leqslant W < N$, $0 \leqslant \rho \leqslant 1$, $N - W < j \leqslant N$ and $i = 1, 2, \ldots, N - W$. Using formulae for the determinant of a partitioned matrix, a closed form expression for the solution of the Markov chain roulette-type game is deduced. The present analysis is supported by two mathematical models from tumor growth and war with bargaining.

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1. Introduction

The classical two-player gambler's ruin problem, in which one unit changes hands until one player goes bankrupt, is well known (cf Feller (1968), chapter 14.8, Srinivasan and Mehata (1978) chapter 3, Percus (1985), El-Shehawey (2008) and Lefebvre (2008) and references therein), has a very simple, closed form solution, given in the cited literature and also referred to in corollary 2.1 of the present paper in the case $\rho = 0$. There are numerous applications of this type of problem, and it attracted some attention in the 1970s in the context of cancer growth modeling (cf Bell (1976) and Beyer and Waterman (1977)). In particular, there was significant interest in estimating the expected time for a cancerous clone of cells to reach the tumor size from a single wayward cell, rather than dying off before reaching the tumor size. The analysis can help to determine the appearance of the first wayward cancerous cell, and perhaps to identify reasons for the cancerous growth (cf El-Shehawey (1994) and references therein). Here, we generalize the classical problem by introducing the following assumptions.

(1) The number of units the gambler may win in each single game is W, $1 \leqslant W < N$.
(2) The initial capital of the gambler is i units, $0 \leqslant i \leqslant N - W$.
(3) The ultimate fortune which the gambler wants to accumulate is N units.
(4) In each single game the gambler has a probability q of winning W units from his adversary, and probability $p = 1 - q$ of losing one unit to his adversary.
(5) If the gambler's fortune drops down to 0 units he is allowed with probability ρ, $0 \leqslant \rho \leqslant 1$, to start the game again with one unit.

(6) After a series of single games, the game ends
   (i) with probability $1 - \rho$ when the gambler goes broke (loses all his fortune), or
   (ii) when he reaches his objective (wins all his adversary's fortune).

In the language of Markov chain random walk—that is, $\{X_n, n = 0, 1, \ldots\}$ whose state space is $\{0, 1, \ldots, N\}$ with transition probabilities

$$p_{i,i+W} = q = 1 - p_{i,i-1} = 1 - p \qquad \text{for} \quad i \in \{1, 2, \ldots, N - W\}.$$

When the particle hits the origin absorption (ruin) partially occurs with probability $1 - \rho$ or reflects to the state one with probability $\rho, 0 \leqslant \rho \leqslant 1$. At the boundary state $j, N - W < j \leqslant N$ there is an absorbing barrier.

We look for the probability that the gambler loses all his fortune, that is the probability of ruin partially

$$P(i) := \Pr\{\tau_0 < \tau_{N-W} \mid X_0 = i\} \qquad \text{for} \quad i \in \{0, 1, 2, \ldots, N - W\},$$

where

$$\tau_0 := \inf\{n \geqslant 0 : X_n = 0\} \qquad \text{and} \qquad \tau_{N-W} := \inf\{n \geqslant 0 : X_n = N - W\}.$$

Physically, this corresponds to a situation where upon reaching the boundary 0, the particle is either lost from the system with probability $1 - \rho$ or turned back to the state 1 of the system with probability ρ, and reduces to the classical problems of random walk, associated with the game of roulette, in the presence of the following.

(1) A perfectly absorbing barrier at $j, N - W < j \leqslant N$ with partially absorbing, perfectly absorbing, or perfectly reflecting barriers at 0 for $0 < \rho < 1, \rho = 0$, and $\rho = 1$, respectively.
(2) A single (partially or perfectly) absorbing barrier at the origin, by taking the limit when $N \to \infty$. It corresponds to a contest in which a gambler plays a game against an infinitely rich adversary.

In the present paper, using formulae for the determinant of a partitioned matrix, we seek a closed form solution to this generalized roulette-type game, i.e., we compute the probability of ruin partially. This problem is solved in several special cases (cf Feller (1968), p 352, Hill and Gulati (1981), Percus (1985) and El-Shehawey (2000)).

Using generating functions and Lagrange's theorem for the expansion of a function as a power series, El-Shehawey (2002) deduced an explicit expression for the probability of the player's capital at the nth step with the assumption that the adversary is infinitely rich ($N \to \infty$) (see also Hill and Gulati (1981)).

Kozek (2002) has derived similar formulae for the complementary probability of ruin, $q(i) := \Pr\{\tau_{N-W} < \tau_0 \mid X_0 = i\}$, in the particular case when $\rho = 0$.

2. Closed form solution for the Markov chain roulette-type game

Let $P(i)$ be the probability of ruin partially for a gambler starting with i units, $0 \leqslant i \leqslant N - W$, with the probability p of losing a unit and $q = 1 - p$ of winning W units and playing against an adversary with initial capital of $N - i$ units. The probability $P(i)$ satisfies the following difference equation

$$P(i) = pP(i-1) + qP(i+W) \qquad \text{for} \quad 1 \leqslant i \leqslant N - W \tag{2.1}$$

with the boundary conditions

$$P(i) = \begin{cases} \rho P(1) + (1-\rho) & \text{for} \quad i = 0 \\ 0 & \text{for} \quad N - W < i \leqslant N. \end{cases} \tag{2.2}$$

Equations (2.1) and (2.2) can be expressed as the following system of $N - W$ equations:

$$
\begin{array}{ll}
P(1) - qP(W+1) = p(\rho P(1) + (1-\rho)), & \text{at} \quad i = 1 \\
-pP(1) + P(2) - qP(W+2) = 0, & \text{at} \quad i = 2 \\
-pP(2) + P(3) - qP(W+3) = 0, & \text{at} \quad i = 3 \\
\qquad\vdots & \qquad\vdots \\
-pP(N-1-2W) + P(N-2W) - qP(N-W) = 0, & \text{at} \quad i = N-2W \\
-pP(N-2W) + P(N-2W+1) = 0, & \text{at} \quad i = N-2W+1 \\
-pP(N-2W+1) + P(N-2W+2) = 0, & \text{at} \quad i = N-2W+2 \\
\qquad\vdots & \qquad\vdots \\
-pP(N-W-1) + P(N-W) = 0, & \text{at} \quad i = N-W.
\end{array}
\tag{2.3}
$$

This system can be written in a matrix form as

$$P_{N-W} Y_{N-W} = R_{N-W}, \tag{2.4}$$

where P_{N-W} is a band matrix of order $N - W$, defined by

$$
P_{N-W} = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ N-2W \\ \\ \vdots \\ \\ N-W \\ \\ \end{array}
\begin{array}{c} 1 \quad 2 \quad \cdots \quad \cdots \quad W+1 \quad \cdots \quad N-W \\
\begin{pmatrix}
1 & 0 & \cdots & 0 & -q & 0 & \cdots & 0 \\
-p & 1 & 0 & & \ddots & -q & \ddots & \vdots \\
0 & -p & & \ddots & & \ddots & \ddots & 0 \\
\vdots & \ddots & & & \ddots & & \ddots & -q \\
\vdots & & \ddots & \ddots & \ddots & \ddots & & 0 \\
\vdots & & & \ddots & & & \ddots & \vdots \\
\vdots & & & & \ddots & -p & 1 & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & -p & 1
\end{pmatrix}\end{array},
\tag{2.5}
$$

with the elements of the matrix $P_{N-W} = (P_{N-W}(i, j))$ being

$$
P_{N-W}(i,j) = \begin{cases}
1, & i = j, \quad j = 1, \ldots, N-W \\
-p, & i = j+1, \quad j = 1, \ldots, N-W-1 \\
-q, & i = j-W, \quad j = W+1, \ldots, N-W \\
0, & \text{otherwise,}
\end{cases}
$$

Y_{N-W} being a column vector of probabilities $P(i), i = 1, 2, \ldots, N - W$, and R_{N-W} being a column vector with the first element equal to $p(\rho P(1) + (1 - \rho))$ and the last $N - W - 1$ elements equal to zero.

Denote

$$\Delta_{N-W} = |P_{N-W}|. \tag{2.6}$$

Lemma 2.1. *Let A be an invertible matrix of order n, partitioned as*

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right],$$

in which A_{22} is non-singular. Then the determinant

$$\det A = \det A_{22} . \det\left(A_{11} - A_{12} A_{22}^{-1} A_{21}\right). \tag{2.7}$$

A matrix with the form of $A_{11} - A_{12}A_{22}^{-1}A_{21}$ *is called a Schur complement of* A_{22} *in* A. *An excellent review of Schur complements and their applications is given by Horn and Johnson (1994), section 07.3, p 18, (see also El-Shehawey et al (2008)).*

Lemma 2.2. *The determinant* Δ_{N-W} *of the matrix* P_{N-W} *obeys the following recursive relation*

$$\Delta_{N-W} = \Delta_{N-W-1} - qp^W \Delta_{N-2W-1}, \tag{2.8}$$

with initial conditions

$$\Delta_1 = \Delta_2 = \cdots = \Delta_W = 1. \tag{2.9}$$

Proof. See the appendix.

Lemma 2.3. *The determinant* $\Delta_{N-W} = |P_{N-W}|$ *can be expressed in the following explicit form*

$$\Delta_{N-W} = \sum_{i=0}^{\lfloor (N-W-1)/(W+1) \rfloor} (-1)^i \binom{N-(i+1)W-1}{j} (qp^W)^i, \tag{2.10}$$

where $\lfloor u \rfloor$ *denotes the greatest integer not exceeding* u.

Proof. See the appendix.

Lemmas 2.2 and 2.3, with appropriate change of notation, agree with that of Kozek (2002).

2.1. Main result

The main purpose of this contribution is to find a closed form solution for the Markov chain roulette-type game, i.e., solving the system of equations (2.4). It is formulated in the next theorem:

Theorem 2.1. *The probability* $P(i)$ *takes the form*

$$P(i) = \begin{cases} \rho P(1) + (1-\rho) & \text{for} \quad i = 0 \\ p^i(\rho P(1) + (1-\rho)) \dfrac{\Delta_{N-W-i}}{\Delta_{N-W}} & \text{for} \quad 1 < i \leqslant N-W \\ 0 & \text{for} \quad N-W < i \leqslant N, \end{cases} \tag{2.11}$$

where

$$P(1) = \frac{p(1-\rho)\Delta_{N-W-1}}{\Delta_{N-W} - \rho p \Delta_{N-W-1}}, \tag{2.12}$$

and Δ_{N-W} *is given by (2.10).*

Proof. (by induction)

Let $P_{N-W}^{(i)}$ denote the matrix obtained from the matrix P_{N-W}, given by (2.5), by replacing its ith column with the column vector R_{N-W} defined in (2.4). We denote by $\Delta_{N-W}^{(i)}$ the determinant of the matrix $P_{N-W}^{(i)}$.

Basis steps. When $i = 1$, partition the matrix $P_{N-W}^{(1)}$ by the first row and the first column in the form

$$
P^{(1)}_{N-W} =
\begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ N-2W \\ \\ \vdots \\ \\ N-W \\ \\ \end{array}
\left(\begin{array}{c|cccccccc}
p(\rho P(1) + (1-\rho)) & 0 & \cdots & 0 & -q & 0 & \cdots & 0 \\
\hline
0 & 1 & 0 & & & -q & & \vdots \\
 & -p & & \ddots & & & \ddots & 0 \\
 & & & & \ddots & & & -q \\
\vdots & & & \ddots & \ddots & \ddots & & 0 \\
 & & & & & & \ddots & \vdots \\
 & & & & & -p & 1 & 0 \\
0 & & \cdots & \cdots & & 0 & -p & 1
\end{array}\right).
\tag{2.13}
$$

Applying lemma 2.1, we obtain

$$\Delta^{(1)}_{N-W} = p(\rho P(1) + (1-\rho))\Delta_{N-W-1}. \tag{2.14}$$

Dividing both sides by Δ_{N-W}, we have

$$\frac{\Delta^{(1)}_{N-W}}{\Delta_{N-W}} = p(\rho P(1) + (1-\rho))\frac{\Delta_{N-W-1}}{\Delta_{N-W}}. \tag{2.15}$$

Using Cramer's rule, we obtain

$$P(1) = \frac{\Delta^{(1)}_{N-W}}{\Delta_{N-W}}. \tag{2.16}$$

Hence the theorem is true for $i = 1$.

Substituting (2.16) into (2.15), we obtain

$$\left(1 - \rho p \frac{\Delta_{N-W-1}}{\Delta_{N-W}}\right) P(1) = p(1-\rho)\frac{\Delta_{N-W-1}}{\Delta_{N-W}}. \tag{2.17}$$

Thus formula (2.12) follows.

Now let $i = 2$. Partition the matrix $P^{(2)}_{N-W}$ by the first two rows and the first two columns in the form

$$
P^{(2)}_{N-W} =
\left(\begin{array}{cc|cccccc}
1 & p(\rho P(1) + (1-\rho)) & \cdots & 0 & -q & 0 & \cdots & 0 \\
-p & 0 & 0 & & & -q & \ddots & \vdots \\
\hline
0 & \vdots & 1 & & & & \ddots & 0 \\
 & & -p & & & & & -q \\
\vdots & & 0 & & & & & 0 \\
\vdots & & \vdots & \ddots & \ddots & \ddots & & \vdots \\
 & & & & \ddots & & & 0 \\
0 & & 0 & \cdots & & 0 & -p & 1
\end{array}\right).
\tag{2.18}
$$

Applying lemma 2.1, we obtain

$$\Delta^{(2)}_{N-W} = p^2(\rho P(1) + (1-\rho))\Delta_{N-W-2}. \tag{2.19}$$

Dividing both sides of (2.19) by Δ_{N-W}, we obtain

$$\frac{\Delta^{(2)}_{N-W}}{\Delta_{N-W}} = p^2(\rho P(1) + (1-\rho))\frac{\Delta_{N-W-2}}{\Delta_{N-W}}. \tag{2.20}$$

Using Cramer's rule again, we have

$$P(2) = \frac{\Delta^{(2)}_{N-W}}{\Delta_{N-W}}. \tag{2.21}$$

Then the theorem is true for $i = 2$ as well.

Induction step. Suppose that the formula (2.11) is true, in the case $1 < i \leqslant N - W$, for $i = j - 1$ and $i = j$, and we try to prove that it holds at $j + W$, i.e. let

$$P(j-1) = p^{j-1}(\rho P(1) + (1-\rho))\frac{\Delta_{N-W-j+1}}{\Delta_{N-W}}, \tag{2.22}$$

$$P(j) = p^{j}(\rho P(1) + (1-\rho))\frac{\Delta_{N-W-j}}{\Delta_{N-W}}. \tag{2.23}$$

From (2.22), (2.23) and the difference equation (2.1) with $i = j$, it is easy to verify that

$$qP(j+W) = \frac{p^j(\rho P(1) + (1-\rho))}{\Delta_{N-W}}\{\Delta_{N-W-j} - \Delta_{N-W-j+1}\}. \tag{2.24}$$

Using lemma 2.2, formula (2.24) finally becomes

$$P(j+W) = p^{j+W}(\rho P(1) + (1-\rho))\frac{\Delta_{N-2W-1}}{\Delta_{N-W}}. \tag{2.25}$$

Hence the theorem holds for $i = j + W$. This completes the proof. □

The following two corollaries are simple consequences of theorem 2.1. In corollary 2.1. The game terminates when the player loses all his fortune or when his capital reaches, or exceeds, level $N - W$.

Corollary 2.1. *Let $P(i)$ be the probability of ruin ($\rho = 0$) for a gambler with an initial capital of i units, $i = 0, 1, \ldots, N$, with the probability p of losing a unit and $q = 1 - p$ of winning W units, and playing against an adversary with initial capital $N - i$ units. Then we have*

$$P(i) = \begin{cases} 1 & \text{for} \quad i = 0 \\ p^i \dfrac{\Delta_{N-W-i}}{\Delta_{N-W}} & \text{for} \quad 1 < i \leqslant N - W \\ 0 & \text{for} \quad N - W < i \leqslant N \end{cases} \tag{2.26}$$

and Δ_{N-W} given by (2.10).

The complementary probability of ruin is

$$q(i) = 1 - P(i) \qquad \text{for} \quad i = 0, 1, \ldots, N, \tag{2.27}$$

where $P(i)$ is given by (2.26).

Formula (2.27), with the appropriate change of notation, agrees with that of Kozek (2002).

Corollary 2.2. *Let $P(i)$ be the probability of ruin partially $(W = 1)$ for the classical gambler's ruin in which the gambler loses or wins one unit with the probabilities p and $q = 1 - p$, respectively. If the gambler's initial capital is i units, $i = 0, 1, \ldots, N - 1$, and he plays against an adversary with initial capital $N - i$ units, then we obtain*

$$P(i) = \begin{cases} \dfrac{(1-\rho)\left[\left(\frac{p}{q}\right)^i - \left(\frac{p}{q}\right)^N\right]}{1 - \rho\frac{p}{q} - (1-\rho)\left(\frac{p}{q}\right)^N} & \text{for} \quad p \neq q \\ \dfrac{N - i}{N + \frac{\rho}{1-\rho}} & \text{for} \quad p = q. \end{cases} \tag{2.28}$$

Proof. Using lemma 2.3, with $W = 1$, it is easy to verify that

$$\Delta_{N-1} = q^{N-1} \sum_{s=0}^{N-1} \left(\frac{p}{q}\right)^s = \frac{q^{N-1}\left(1 - \left(\frac{p}{q}\right)^N\right)}{1 - \frac{p}{q}}, \tag{2.29}$$

$$\Delta_{N-i-1} = \frac{q^{N-i-1}\left(1 - \left(\frac{p}{q}\right)^{N-i}\right)}{1 - \frac{p}{q}} \quad \text{and} \quad \Delta_{N-2} = q^{N-2}\left(\frac{1 - \left(\frac{p}{q}\right)^{N-1}}{1 - \frac{p}{q}}\right). \tag{2.30}$$

Substituting (2.29) and (2.30) into theorem 2.1, with $W = 1$, after simplification, formula (2.28) follows in the case $p \neq q$. The case $p = q$ immediately follows from the case $p \neq q$ after taking the limit as $\frac{p}{q} \to 1$. □

It is straightforward to show that the present results are generalized versions of some well-known results. For example, formula (2.28), with appropriate change of notation, agrees with that of El-Shehawey (2000) (see also Percus (1985)), agrees in the case $\rho = 0$, $N \to \infty$ with that of El-Shehawey (2002), and agrees in the case $\rho = 0$ with that of Lefebvre (2008) (see also Feller (1968) p 345, and El-Shehawey (1994)).

Applications

The gambler's ruin problem has found applications in biology (especially, tumor growth and genetics), in physics (especially, solar physics, solid state physics and quantum mechanics), in war with bargaining and others (cf Watterson (1961), Beyer and Waterman (1979), Fokker (1984), Bohm (1994), Duan and Howard (1996), Konno (2002), and Smith and Stam (2004)). For example we consider two mathematical models, the first one to which our result in formula (2.28) has been applied concerns the growth of a cancer tumor. A tumor is a non-inflamed abnormal mass of tissue. A cancer tumor is thought of, in a simple model, as arising from one wayward cell that has lost the ability to control itself. The wayward cell starts reproducing without regard to the presence of other cells. The cell progeny may all die before catastrophe overtakes the host organism, or may produce a family tree of descendants (called a clone) large enough to be noticeable to the host organism, in which case the descendants become a tumor. It is the tumor that is noticeable, not the early cells that died without progeny. It is of interest to estimate the time it takes for one wayward cell to reach the tumor size.

A set of models has been developed (see for example Bell (1976) and El-Shehawey (1994)). Consider a process in which each cell has probability $p(0 < p < 1)$ of dividing to produce two identical new cells, and probability $q = 1 - p$ of dying or of becoming irreversibly changed to a non-dividing state. For normal cells in a tissue that is not undergoing net growth, $p = \frac{1}{2}$. Of interest is the event of a cell becoming, by this mechanism, a macroscopic clone of N cells. To complete this model, division and death times must be specified. The question of interest is the time it takes for one wayward cell to reach a clone of N cells. The natural model for this problem is a birth and death process with linear growth. The complementary probability of ruin, that a single cell ever becomes a clone of N proliferating cells is then found as the solution of a gambler's ruin problem, $q(i) = 1 - P(i)$ where $P(i)$ is given by (2.28).

A second model arises from a war with bargaining. The present roulette-type game to which the general formula (2.11) has been applied concerns a war with bargaining, the simplest case $W = 1$ and $\rho = 0$ was suggested by Smith and Stam (2004). For the sake of illustration, assume the existence of two nations of different military powers, A_1 and A_2, that are in dispute over the division of some prize. Assume that nation A_1 is stronger. Suppose there are N forts and that initially nation A_1 possesses X_0 of them. Each time the nations fight, either A_1 captures W forts from A_2, or A_2 captures a fort from A_1. These events occur with probability q and $p = 1 - q$, respectively. If A_1 wins the first battle, then the distribution of forts becomes $X_1 = X_0 + W$; similarly, if A_1 loses the first battle, then $X_1 = X_0 - 1$. We refer to the distribution of forts as the 'state' variable. The war terminates when either the total capture of the nation A_1 for the first time reaches or exceeds $N - W$ forts, or its fortune drops down to 0 forts with probability $1 - \rho$. The war can break out again, with probability ρ, if the loser can get some aid (a fort) from another nation. The probability that nation A_1 will lose partially the war given the starting state i is given by theorem 2.1. The $(N - W) \times (N - W)$ transition probability matrix P_{N-W}, given in (2.5), describes movements between states. For example, the element ij is the probability of moving from state i to state j.

3. Conclusion

In this paper, we have considered the so-called Markov chain roulette-type game that generalizes the classical gambler's ruin problem, in which at 0 the particle is either absorbed with probability $1 - \rho$ or reflected to the position one with probability ρ. And at $j, N - W < j \leqslant N$ there is an absorbing barrier. At each step the particle either moves a fixed multiple displacement to the right or a unit displacement to the left. The present results provide a simple and convenient approach to finding an exact and explicit expression for the probability of ruin partially. The expression we reported has been derived by means of an elementary partitioned matrix. The present formula enables us to find interesting particular cases through an appropriate choice of the reflection probability ρ and the number of units W. We have discussed two applications in tumor growth and war with bargaining.

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Appendix

Proof of lemma 2.2. Expanding the determinant Δ_{N-W} of the matrix given in (2.5) by the elements of the first row, we obtain

$$\Delta_{N-W} = \Delta_{N-W-1} - (-1)^W(-q)$$

$$
\times \left|\begin{array}{cccc|cccccccc}
-p & 1 & 0 & \cdots & -q & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & 0 & & \ddots & & & & & \vdots \\
\vdots & & \ddots & 1 & & & \ddots & & & & \vdots \\
0 & \cdots & 0 & -p & & & & -q & 0 & \cdots & 0 \\
\hline
0 & \cdots & \cdots & 0 & 1 & 0 & \cdots & 0 & -q & 0 & \vdots \\
 & & & & -p & 1 & \ddots & & & \ddots & 0 \\
\vdots & & & \vdots & 0 & -p & \ddots & \ddots & & & -q \\
\vdots & & & \vdots & & & \ddots & \ddots & \ddots & & 0 \\
\vdots & & & \vdots & \vdots & & & \ddots & \ddots & \ddots & \vdots \\
 & & & & & & & & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & 0 & & \cdots & & 0 & -p & 1
\end{array}\right|. \tag{A.1}
$$

Applying lemma 2.1, we obtain

$$
\Delta_{N-W} = \Delta_{N-W-1} + (-1)^{W}(-q) \times \left|\begin{array}{ccccc}
-p & 1 & 0 & \cdots & 0 \\
0 & & & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
\vdots & & & & 1 \\
0 & \cdots & \cdots & 0 & -p
\end{array}\right| \left|\begin{array}{cccccccc}
1 & 0 & \cdots & 0 & -q & 0 & \cdots & 0 \\
-p & 1 & 0 & & \ddots & -q & \ddots & \vdots \\
0 & -p & 1 & \ddots & & & \ddots & 0 \\
 & & & & \ddots & & & -q \\
\vdots & & & \ddots & \ddots & \ddots & & 0 \\
\vdots & & & & & & \ddots & \vdots \\
\vdots & & & & & -p & 1 & 0 \\
0 & \cdots & \cdots & \cdots & & 0 & -p & 1
\end{array}\right|, \tag{A.2}
$$

and formula (2.8) immediately follows. □

Proof of lemma 2.3. (by induction)

Using lemma 2.2, it is clear that for $N - W = 1$, $\Delta_1 = 1$.

Assume that lemma 2.3 is valid for $N - W - 1, N - W - 2, \ldots, N - 2W - 1$ for every $N - W \geqslant 2$. We will show that this implies its validity for $N - W$. From formula (2.8) we have

$$
\Delta_{N-W} = \sum_{i=0}^{\lfloor (N-W-2)/(W+1) \rfloor} (-1)^i \binom{N - W - iW - 2}{i} (qp^W)^i - qp^W \sum_{i=0}^{\lfloor (N-2W-2)/(W+1) \rfloor} (-1)^i \binom{N - 2W - iW - 2}{i} (qp^W)^i
$$

$$= \sum_{i=0}^{\lfloor (N-W-2)/(W+1) \rfloor} (-1)^i \binom{N-W-iW-2}{i} (qp^W)^i$$
$$+ \sum_{i=0}^{\lfloor (N-W-1)/(W+1)-1 \rfloor} (-1)^{i+1} \binom{N-2W-iW-2}{i} (qp^W)^{i+1}$$
$$= 1 + \sum_{i=1}^{\lfloor (N-W-2)/(W+1) \rfloor} (-1)^i \binom{N-W-iW-2}{i} (qp^W)^i$$
$$+ \sum_{i=1}^{\lfloor (N-W-1)/(W+1) \rfloor} (-1)^i \binom{N-W-iW-2}{i-1} (qp^W)^i, \qquad \text{(A.3)}$$

and formula (2.10) immediately follows if the integer parts of $(N-W-1)/(W+1)$ and $(N-W-2)/(W+1)$ are equal, using the fact $\binom{N-(i+1)W-2}{i} + \binom{N-(i+1)W-2}{i-1} = \binom{N-(i+1)W-1}{i}$.

Note that there is one, last term in the last sum in (A.3), not matched, if the former integer is strictly smaller than the latter one, i.e.

$$\lfloor (N-W-2)/(W+1) \rfloor < \lfloor (N-W-1)/(W+1) \rfloor \quad \text{or}$$
$$\lfloor (N-W-1)/(W+1) - 1/(W+1) \rfloor < \lfloor (N-W-1)/(W+1) \rfloor.$$

This happens only when $N-W-1$ is a multiple of $W+1$, say $N-W = k(W+1)+1$. Then the last value of index i in the last sum in (A.3) is equal to k. Hence, the coefficient of $(qp^W)^k$ is $(-1)^k \binom{k(W+1)-1-kW}{k-1} = (-1)^k \binom{k-1}{k-1} = (-1)^k$, and the not-matched term of the last sum in (A.3) coincides with the last term of (2.10). Consequently formula (2.10) holds true. □

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